On gradient enriched elasticity theories: A reply to "Comment on 'On non-singular crack fields in Helmholtz type enriched elasticity theories' " and important theoretical aspects

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Abstract

The Comment by Aifantis that criticizes the article 'On non-singular crack fields in Helmholtz type enriched elasticity theories' [Lazar, M., Polyzos, D., 2014. Int. J. Solids Struct. doi: 10.1016/j.ijsolstr.2014.01.002] is refuted by means of clear and straightforward arguments. Important theoretical aspects of gradient enriched elasticity theories which emerge in this work are also discussed.

Keywords: Cracks, dislocations, fracture mechanics, nonlocal elasticity, gradient elasticity

1. Introduction

A comment by Aifantis (2014) [thereafter referred to as (A)] criticizes some aspects of the article (Lazar and Polyzos, 2014) [referred to as (LP)]. In the (LP) article, we discuss and point out the physical or unphysical meaning and interpretation of simple non-singular stress fields of cracks of mode I and mode III published by Aifantis (2009, 2011); Isaksson et al. (2012) and Isaksson and Hägglund (2013). In particular, we investigate if the aforementioned solutions satisfy the equilibrium conditions, boundary conditions and compatibility conditions in the framework of nonlocal or gradient elasticity theory. In the meantime, Konstantopoulos and Aifantis (2013) and Isaksson and Dumont (2014) have continued to publish their non-singular stress fields in other articles in scientific journals. The aim of the (LP) article was to show that the results of (Aifantis, 2009, 2011; Isaksson et al., 2012; Isaksson and Hägglund, 2013) cannot be the correct solutions of a nonlocal or gradient compatible elastic fracture mechanics problem, since important physical conditions are violated and not to discuss about the physical assumptions of the GRADELA model made by Aifantis and others. In this work, we reply to the comments of (A) which are relevant to the (LP) article and we clarify important aspects of gradient enriched elasticity theories.

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2. Discussion

We review in the (LP) article the framework of Eringen's nonlocal elasticity theory (Eringen, 1983, 2002). In this context, (A) claimed that "Eringen never uses the classical stress σ^0_{ij} in a balance law[1] $\sigma^0_{ij,j}=0$", writing additionally that "Eq. (LP5) is an artifact of a number of assumptions and not a fundamental equation of Eringen's theory". It is easy to show that (A) is mistaken concerning this point due to the following reasons. First of all, if the tensor σ^0_{ij} is the classical stress tensor, then it has to fulfill certainly the classical equilibrium condition $\sigma^0_{ij,j}=0$. Moreover, combining Eq. (LP4) with Eq. (LP7), one finds

$$Lt_{ij,j}=\sigma^0_{ij,j}=0\,, \tag{1}$$

which proves that the relation $\sigma^0_{ij,j}=0$ is true and is part of Eringen's theory of nonlocal elasticity. Here,

$$L=1-\ell^2\Delta \tag{2}$$

is the characteristic differential operator appearing in Eringen's nonlocal elasticity and in the corresponding gradient elasticity, t_{ij} is the stress tensor of nonlocal elasticity, ℓ is a characteristic length scale and Δ denotes the Laplacian. L is usually called as Helmholtz operator in gradient enriched elasticity theories (see, Unger and Aifantis (1995); Vardoulakis et al. (1996); Gutkin and Aifantis (1999); Peerlings et al. (2001); Paulino et al. (2003); Gutkin and Ovid'ko (2004); Lazar and Maugin (2005);

[1] Actually, $\sigma^0_{ij,j}=0$ is a conservation law.

Chan et al. (2006); Aifantis (2009); Lazar (2013)). In general, the Helmholtz equation $\Delta u + cu = 0$, $c =$ constant, can be found with positive ($c > 0$) or negative sign ($c < 0$) in the mathematical literature (e.g., Koshlyakov et al. (1964); Roos (1969); Polyanin (2001); Hassani (2002)); and only sometimes the Helmholtz equation with negative sign, that is when $c < 0$, is referred to as the modified Helmholtz equation (e.g., Zauderer (1983)). Moreover, Eringen (1983, 2002) stated that in the static case of nonlocal elasticity, the classical equilibrium condition (Eq. (LP5)) is to be satisfied (see, e.g., Eq. (6.9.32) in Eringen (2002)). Thus, the classical equilibrium condition is part of Eringen's theory of nonlocal elasticity in contrast to the claim of (A). In addition, we note that anisotropic nonlocal elasticity of generalized Helmholtz type can be found in Lazar and Agiasofitou (2011).

Afterwards, we review in (LP) a simplified gradient elasticity theory, the so-called gradient elasticity of Helmholtz type. This simplified and straightforward version of gradient elasticity with only one length scale ℓ is a particular case of Mindlin's theory of gradient elasticity of form II (Mindlin, 1964) and it has been called by Lazar et al. (2005) "gradient elasticity of Helmholtz type". The development of gradient elasticity of Helmholtz type (Lazar and Maugin, 2005) was strongly influenced and stimulated by the field theoretical framework of the so-called dislocation gauge theory (Lazar, 2002a,b, 2003a,b) and by Feynman's lectures on gravitation (Feynman, 1995) (see also Lazar and Anastassiadis (2009); Agiasofitou and Lazar (2010)). As shown in Lazar and Maugin (2005); Lazar (2013, 2014), gradient elasticity of Helmholtz type is based on proper variational principles including double stresses and serves a "physical" regularization based on higher-order partial differential equations. In Lazar and Maugin (2005, 2006); Lazar (2012, 2013, 2014) gradient elasticity of Helmholtz type has been successfully applied to calculate non-singular stresses, non-singular strains and non-singular displacement fields of straight dislocations and dislocation loops. It is noted that Lazar and Maugin (2005) is the first work where such a task was undertaken within the framework of Mindlin's gradient elasticity with the remarkable outcome that both the stress and the strain singularities are removed at the dislocation line in contrast to the corresponding solutions of classical elasticity. Moreover, Lazar and Maugin (2006) corrected the mistaken gradient elasticity solutions for the displacement fields of straight screw and edge dislocations given by Gutkin and Aifantis (1996, 1997, 1999). Later, (A) adopted the correct "Lazar-Maugin solutions" in further studies (Davoudi and Aifantis, 2013) (see, e.g., Gutkin (2006); Davoudi et al. (2010)). It should be mentioned that the non-singular dislocation key-formulas obtained in gradient elasticity of Helmholtz type (Lazar, 2013, 2014) have been recently implemented in 3D discrete dislocation dynamics (Po et al., 2014). The J-integral was given by Lazar and Kirchner (2007) for incompatible gradient elasticity of Mindlin and Helmholtz type and by Agiasofitou and Lazar (2009) for Mindlin's gradient elasticity of grade three. In addition, Lazar (2014) shows that the framework of gradient elasticity of Helmholtz type is in agreement with the framework of the magnetostatic gradient theory, which is the magnetostatic part of the so-called Bopp-Podolsky theory (Bopp, 1940; Podolsky, 1942) and how gradient theories are used in physics. However, it has to be mentioned that Mindlin (1964) introduced his theory of gradient elasticity without giving any credit to Bopp and Podolsky. Günther (1976) was the first to speak of a mechanical model of the Bopp-Podolsky potential for defects in elasticity.

On the other hand, about three decades after Mindlin (1964), Aifantis (1992); Altan and Aifantis (1992); Ru and Aifantis (1993); Gutkin and Aifantis (1996, 1997) introduced a simple gradient elasticity model called GRADELA. In fact, Aifantis (1992) has intuitively postulated a gradient modification of Hooke's law. However, the framework of Aifantis (1992); Altan and Aifantis (1992); Gutkin and Aifantis (1996, 1997) lacks double stresses and, consequently, the stress tensor remains singular. Their approach postulates a sophisticated Hooke's law including a Laplacian of the elastic strain tensor. In a refined version of their model, Gutkin and Aifantis (1999) found non-singular stress and non-singular strain fields using a more sophisticated Hooke's law including a Laplacian of the (Cauchy) stress tensor. Moreover, in contrast to Mindlin's theory of gradient elasticity (Mindlin, 1964) (see also Jaunzemis (1967)), the framework of Aifantis (1992); Altan and Aifantis (1992) and Gutkin and Aifantis (1996, 1997, 1999) is not based on proper variational considerations (e.g. to obtain pertinent boundary conditions). Thus, GRADELA is based on ad-hoc postulations and not on a proper theoretical framework as basis of a theory (see also Fafalis et al. (2012) for an interesting discussion). It becomes evident that gradient elasticity of Helmholtz type is different from GRADELA due to the aforementioned reasons and that the authors of the (LP) article have not adopted GRADELA.

Now, we sketch the basics of Mindlin's theory of gradient elasticity of form II (Mindlin, 1964) and we give its relation to the theory of gradient elasticity of Helmholtz type. In Mindlin's theory of gradient elasticity of form II, the strain energy density has the general form

$$W = W(e_{ij}, e_{ij,k})\,, \tag{3}$$

where e_{ij} denotes the elastic strain tensor of gradient elasticity. The stresses of gradient elasticity are defined as (see, also Eshel and Rosenfeld (1970, 1973))

$$\tau_{ij} \equiv \frac{\partial W}{\partial e_{ij}} = \tau_{ji} \qquad \text{Cauchy stress tensor}\,, \tag{4}$$

$$\tau_{ijk} \equiv \frac{\partial W}{\partial e_{ij,k}} = \tau_{jik} \qquad \text{double stress tensor}\,. \tag{5}$$

Thus, τ_{ij} is the Cauchy stress tensor of gradient elasticity which usually reads

$$\tau_{ij} = C_{ijkl} e_{kl} \,, \tag{6}$$

where C_{ijkl} is the tensor of the elastic constants. In the (LP) article, we use the notation Cauchy-like stress tensor for τ_{ij}, which means Cauchy stress tensor of gradient elasticity. The equilibrium condition for the stresses takes the form (Mindlin, 1964)

$$\tau_{ij,j} - \tau_{ijk,kj} = 0 \,. \tag{7}$$

The relation of Mindlin's theory of gradient elasticity of form II to gradient elasticity of Helmholtz type is given by the particular form of the double stress tensor (5)

$$\tau_{ijk} = \ell^2 \tau_{ij,k} \,. \tag{8}$$

The double stress tensor (8) is nothing but the gradient of the Cauchy stress tensor of gradient elasticity multiplied by the pre-factor ℓ^2 in order to have the correct dimension (see, e.g., Lazar and Maugin (2005); Polyzos et al. (2003)). Eq. (8) can be easily obtained by Eq. (5) using the strain energy density of Mindlin's theory of Form II (see Eq. (11.3) in Mindlin (1964)) with an appropriate simplification of the five gradient coefficients (see, e.g., Lazar and Maugin (2005)). The equilibrium condition (7) with the help of the relation (8) can be reduced to

$$L\tau_{ij,j} = 0 \,, \tag{9}$$

where L is the Helmholtz operator given in Eq. (2). Eq. (9) is the equilibrium condition in gradient elasticity of Helmholtz type. Furthermore, substituting Eq. (6) and a compatible elastic strain tensor $e_{ij} = 1/2(u_{i,j} + u_{j,i})$ into Eq. (9), one can find a homogenous Helmholtz-Navier equation for the displacement vector u_k

$$LC_{ijkl} u_{k,lj} = 0 \,, \tag{10}$$

which is an elliptic partial differential equation of fourth-order.

Using an operator split (see, also Vekua (1967); Lazar (2014); Ru and Aifantis (1993)), one can decompose the partial differential equation of third-order (9) into a system of partial differential equations consisting of a partial differential equation of first-order and a partial differential equation of second-order, as follows

$$\sigma^0_{ij,j} = 0 \,, \tag{11}$$

$$L\tau_{ij} = \sigma^0_{ij} \,. \tag{12}$$

Eq. (12) is an inhomogeneous Helmholtz equation where the inhomogeneous part is given by σ^0_{ij}, which has to fulfill Eq. (11). Since σ^0_{ij} satisfies the classical equilibrium condition (11) and does not depend on the length scale ℓ, it may be identified with the Cauchy stress tensor of classical elasticity theory (see also Lazar (2014)). If one substitutes Eq. (6) into Eq. (12), then one obtains

$$LC_{ijkl} e_{kl} = \sigma^0_{ij} \,, \tag{13}$$

which is a partial differential equation for the elastic strain tensor of gradient elasticity of Helmholtz type. Eq. (13) should not be confused with a kind of modified Hooke's law; what is done in the "basic postulate" of GRADELA (Aifantis, 1992; Altan and Aifantis, 1992; Gutkin and Aifantis, 1996, 1997).

One of the main results of the (LP) article (what (A) says as the first major criticism of the (LP)) is that: "Eqs. (30) and (31) are not solution for a mode III crack in nonlocal elasticity since they do not satisfy the equilibrium condition (4)". This is commented by (A) in the framework of gradient elasticity and not in the framework of nonlocal elasticity causing great confusion to the reader. Actually, the comment takes place in the wrong framework. We would like to mention again that the non-singular stresses of mode III, Eqs. (LP30) and (LP31), as well as of mode I, Eqs. (LP47)–(LP49), do not satisfy the equilibrium condition (LP4) of nonlocal elasticity. Thus, the mentioned nonlocal stresses cannot be considered as physical solutions in nonlocal elasticity, since they produce forces at the corresponding crack tips as shown in Eq. (LP35) for mode III and in Eqs. (LP51) and (LP52) for mode I.

Another important result of the (LP) article (what (A) says as the second major criticism of the (LP)) is that the non-singular stresses and the corresponding non-singular strains of mode III and mode I do not fulfill the compatibility conditions in the framework of gradient elasticity and they give rise to non-vanishing dislocation density contributions at the crack tips. In particular, (A) comments the fact that the strains (33) and (34) exhibited in (LP) article (originally published in Aifantis (2009)) are not compatible by writing that the strains (46) in Lazar and Maugin (2005) are also not compatible; stating moreover erroneously that there is a contradiction between (LP) article and Lazar and Maugin (2005) concerning this point. It should be here clarified that there is no contradiction, since there is a great difference between the two problems. Lazar and Maugin (2005) investigate a problem of dislocations in the framework of incompatible gradient elasticity theory, where the strains are incompatible. Whereas, Aifantis (2009, 2011); Isaksson et al. (2012) and Isaksson and Hägglund (2013) investigate a problem of cracks and starting from classical compatible elastic distortions, they obtain incompatible elastic distortions due to gradient elasticity. In other words, gradient elasticity creates incompatibilities (dislocations) in such a problem of cracks as an artifact.

Furthermore, it is misleading to introduce "microstress" and "macrostress" in gradient elasticity as done in (A). Concepts of "microstress" and "macrostress" make sense in microcontinuum theories like micromorphic elasticity, where microstrain is defined and contained (see, e.g., Eringen (1999)).

Moreover, we show that the fact that the stresses (6) and (8) are divergence-free, that is $\tau_{ij,j} = 0$ and $\tau_{ijk,kj} = 0$, is an outcome of the theory and it is not restrictive as it is claimed by (A). These properties of the stresses were first pointed out by Lazar and Maugin (2005). Let us discuss first some properties of the solution of the inhomogeneous Helmholtz equation (12) or (LP19) appearing in gradient elasticity of Helmholtz type. Using the theory of partial differential equations (see, e.g., Vladimirov (1971)), we obtain the representation of the stress tensor τ_{ij} as convolution of the Green function G of the Helmholtz equation, $LG = \delta$, with the classical singular stress tensor σ^0_{ij}

$$\tau_{ij} = G * \sigma^0_{ij}\,, \tag{14}$$

which is the (particular) solution of the inhomogeneous Helmholtz equation (12). δ is the Dirac delta function and $*$ denotes the spatial convolution. As discussed in Lazar (2014), the tensor τ_{ij} has the physical meaning of the Cauchy stress tensor of gradient elasticity; an interpretation which is in full agreement with the interpretation used by Eshel and Rosenfeld (1970, 1973); Gao and Ma (2010a,b); Ma and Gao (2011). By applying the Helmholtz operator L to both sides of Eq. (14), the result reads (see, e.g., Kanwal (1983))

$$L\tau_{ij} = L(G * \sigma^0_{ij}) = (LG) * \sigma^0_{ij} = \delta * \sigma^0_{ij} = \sigma^0_{ij}\,, \tag{15}$$

which proves that Eq. (14) solves Eq. (12). Such solution is unique in the class of generalized functions. If we use Eq. (14), the property of the differentiation of a convolution and that the operation of convolution is commutative (Vladimirov, 1971; Kanwal, 1983), then we find that the divergence of the Cauchy stress tensor of gradient elasticity is zero

$$\tau_{ij,j} = (G * \sigma^0_{ij})_{,j} = G * (\sigma^0_{ij,j}) = 0\,, \tag{16}$$

since $\sigma^0_{ij,j} = 0$ (see Eq. (11)). Hence, $\tau_{ij,j} = 0$. Next, we show that $\tau_{ijk,kj} = 0$. In order to differentiate a convolution, it suffices to differentiate any one of the factors (Kanwal, 1983). Therefore, if the convolution (14) exists, then the Cauchy stress tensor of gradient elasticity is self-equilibrated. If one uses a so-called "Bifield" ansatz for the stress (Lazar, 2014)

$$\tau_{ij} = \sigma^0_{ij} + \sigma^1_{ij}\,, \tag{17}$$

where the tensor σ^0_{ij} is the classical stress tensor and the tensor σ^1_{ij}, which is the gradient part of the stress, corresponds to the relative stress tensor (see Lazar (2014)), then $\sigma^1_{ij,j} = 0$ follows from Eq. (16). Moreover, the double stress tensor (8) (see Eq. (LP11)) is equilibrated by the relative stress tensor σ^1_{ij} (see also Vardoulakis and Sulem (1995))

$$\sigma^1_{ij} = \tau_{ijk,k} = \ell^2 \Delta \tau_{ij}\,. \tag{18}$$

It is easy to see that the following relation holds

$$\sigma^1_{ij,j} = \tau_{ijk,kj} = \ell^2 \Delta \tau_{ij,j} = 0\,, \tag{19}$$

but $\tau_{ijk,k} \neq 0$. Therefore, we have shown that the stresses τ_{ij} and τ_{ijk} are divergence-free, $\tau_{ij,j} = 0$ and $\tau_{ijk,kj} = 0$, and this result arises naturally from the theory. Note that the non-singular gradient-enriched stresses of dislocations and disclinations given by Gutkin and Aifantis (1999, 2000) are also divergence-free. Additionally, using Eq. (17), the boundary conditions of gradient elasticity in terms of σ^0_{ij} and σ^1_{ij} can be found in Aravas (2011) and Lazar (2014).

The $r^{-3/2}$ behavior of the "total" stresses at the crack tip in the framework of strain gradient elasticity is known since 2000; confirmed theoretically (e.g, Aravas and Giannakopoulos (2009); Fannjiang et al. (2002); Georgiadis (2003); Gourgiotis and Georgiadis (2009); Gourgiotis et al. (2010); Shi et al. (2000); Stamoulis and Giannakopoulos (2008)) and numerically (e.g., Akarapu and Zbib (2006); Karlis et al. (2007); Markolefas et al. (2008); Papanicolopulos et al. (2010)). Most of these papers should be known to Aifantis for many years (see, e.g., in Aifantis (2009, 2011)). An appropriate treatment of a crack problem in the framework of strain gradient elasticity is to solve the homogeneous Helmholtz-Navier equation for the displacement vector (Eq. (10)), since a crack problem is a compatible elasticity problem. Moreover, Eq. (10) has to be accompanied by the corresponding boundary conditions of gradient elasticity, the so-called Mindlin boundary conditions (see Eqs. (LP10)) which are stemming from variational principles, and by the appropriate regularity conditions. One can see clearly this procedure in Georgiadis (2003) for a mode III crack problem. Then, the "total" stress becomes singular with $r^{-3/2}$ singularity at the crack tip, as it was predicted in the aforementioned theoretical and numerical works. In addition, Gourgiotis and Georgiadis (2009) solved the homogeneous Helmholtz-Navier equation (10) of a compatible elasticity problem for mode I and mode II cracks together with the appropriate boundary and regularity conditions without using the "Ru-Aifantis theorem". They found that even if the "total" stress is hypersingular, the "monopolar" stress tensor which is directly connected with the compatible elastic strain tensor (and corresponds to the Cauchy stress tensor of gradient elasticity in our notation) is bounded in the vicinity of the crack tip. Moreover, the strain field is also bounded at the crack tip region.

Later, (A) writes about his students and very personal views. In the rest of his comment (A) discusses some unpublished results which are related to forthcoming papers and not to the results of (A) criticized in the (LP) article. Therefore, there is no need to discuss these items.

3. Conclusion

The criticisms of (A) against results in (LP) are refuted by means of clear and straightforward arguments from the mathematical and physical point of view. The

present discussion reveals also that consistency is of major significance, since the two comments of (A) against the two major results of the (LP) article take place in the wrong framework; confusing and mixing nonlocal elasticity with gradient elasticity and compatible elasticity with incompatible elasticity. A problem of cracks has a different treatment than a problem of dislocations.

Therefore, we conclude again that the non-singular stresses given by Isaksson et al. (2012) cannot be considered as physical crack solutions in nonlocal elasticity and the non-singular strains given by Aifantis (2009, 2011) and Isaksson and Hägglund (2013) cannot be considered as physical crack solutions in strain gradient elasticity. In nonlocal elasticity the equilibrium condition is not satisfied, and in gradient elasticity the compatibility condition is not fulfilled. In addition, the non-singular solutions of Aifantis (2011); Isaksson et al. (2012); Isaksson and Hägglund (2013) for a mode I crack do not satisfy the zero stress boundary condition at the crack faces.

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References

Agiasofitou, E.K., Lazar, M., 2009. Conservation and balance laws in linear elasticity of grade three. Journal of Elasticity **94**, 69–85.

Agiasofitou, E., Lazar, M., 2010. On the nonlinear continuum theory of dislocations: a gauge field theoretical approach. Journal of Elasticity **99**, 163–178.

Aifantis, E.C., 1992. On the role of gradients in the localization of deformation and fracture. International Journal of Engineering Science **30**, 1279–1299.

Aifantis, E.C., 2009. On scale invariance in anisotropic plasticity, gradient plasticity and gradient elasticity. International Journal of Engineering Science **47**, 1089–1099.

Aifantis, E.C., 2011. A note on gradient elasticity and nonsingular crack fields. Journal of the Mechanical Behavior of Materials **20**, 103–105 (published June 2012).

Aifantis, E.C., 2014. Commentary on 'On non-singular crack fields in Helmholtz type enriched elasticity theories', International Journal of Solids and Structures, this issue.

Akarapu, S., Zbib, H.M., 2006. Numerical analysis of plane cracks in strain-gradient elastic materials. International Journal of Fracture **141**, 403–430.

Altan, B.C., Aifantis, E.C., 1992. On the structure of the mode III crack-tip in gradient elasticity. Scripta Metallurgica et Materialia **26**, 319–324.

Aravas, N., 2011. Plane-strain problems for a class of gradient elasticity models–A stress function approach. Journal of Elasticity **104**, 45–70.

Aravas, N., Giannakopoulos, A.E., 2009. Plane asymptotic crack-tip solutions in gradient elasticity. International Journal of Solids and Structures **46**, 4478–4503.

Bopp, F., 1940. Eine lineare Theorie des Elektrons. Annalen der Physik (Leipzig) **38**, 345–384.

Chan, Y.-S., Paulino, G.H., Fannjing, A.C., 2006. Change of constitutive relations due to interaction between strain-gradient effect and material gradation. ASME Journal of Applied Mechanics **73**, 871–875.

Davoudi, K.M., Gutkin, M.Yu., Shodja, H.M., 2010. A screw dislocation near a circular nano-inhomogeneity in gradient elasticity. International Journal of Solids and Structures **47**, 741–750.

Davoudi, K.M., Aifantis, E.C., 2013. Nanomechanics of a screw dislocation in a functionally graded material using the theory of gradient elasticity. Journal of the Mechanical Behavior of Materials **21**, 187–194.

Eringen, A.C., 1983. On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. Journal of Applied Physics **54**, 4703–4710.

Eringen, A.C., 1999. Microcontinuum Field Theories I: Foundations and Solids. New York: Springer.

Eringen, A.C., 2002. Nonlocal Continuum Field Theories. New York: Springer.

Eshel, N.N., Rosenfeld, G., 1970. Effects of strain-gradient on the stress-concentration at a cylindrical hole in a field of unaxial tension. Journal of Engineering Mathematics **4**, 97–111.

Eshel, N.N., Rosenfeld, G., 1973. Some Two-Dimensional Exterior Problems in a Linear Elastic Solid of Grade Two. Journal of Applied Mathematics and Mechanics (ZAMM) **53**, 761–772.

Fafalis, D.A., Filopoulos, S.P., Tsamasphyros, G.J., 2012. On the capability of generalized continuum theories to capture dispersion characteristics at the atomic scale. European Journal of Mechanics A/Solids **36**, 25–37.

Fannjiang, A.C., Chan, Y.-C., Paulino, G.H., 2002. Strain Gradient Elasticity for Antiplane Shear Cracks: A Hypersingular Integrodifferential Equation Approach. SIAM Journal of Applied Mathematics **62**, 1066–1091.

Feynman, R.P., 1995. Lectures on Gravitation. Lecture notes by F.B. Morinigo and W.G. Wagner (California Institutes of Technology, Pasadena, California 1962/63), Addison-Wesley.

Gao, X.-L., Ma, H.M., 2010a. Solution of Eshelby's inclusion problem with a bounded domain and Eshelby's tensor for a spherical inclusion in a finite matrix based on a simplified strain gradient elasticity theory. Journal of the Mechanics and Physics of Solids **88**, 779–797.

Gao, X.-L., Ma, H.M., 2010b. Strain gradient solution for Eshelby's ellipsoidal inclusion problem. Proceedings of the Royal Society A **466**, 2425–2446.

Georgiadis, H.G., 2003. The mode III crack problem in microstructured solids governed by dipolar gradient elasticity. ASME Journal of Applied Mechanics **70**, 517–530.

Gourgiotis, P.A., Georgiadis, H.G., 2009. Plane-strain crack problems in microstructured solids governed by dipolar gradient elasticity. Journal of the Mechanics and Physics of Solids **57**, 1898–1920.

Gourgiotis, P.A., Sifnaiou, M.D., Georgiadis, H.G., 2010. The problem of sharp notch in microstructured solids governed by dipolar gradient elasticity. International Journal of Fracture **166**, 179–201.

Günther, H., 1976. On a mechanical model of the Bopp-Podolsky potential. Annalen der Physik (Leipzig) **33**, 448–454.

Gutkin, M.Yu., Aifantis, E.C., 1996. Screw dislocation in gradient elasticity. Scripta Materialia **35**, 1353–1358.

Gutkin, M.Yu., Aifantis, E.C., 1997. Edge dislocation in gradient elasticity. Scripta Materialia **36**, 129–135.

Gutkin, M.Yu., Aifantis, E.C., 1999. Dislocations in the theory of gradient elasticity. Scripta Materialia **40**, 559–566.

Gutkin, M.Yu., Aifantis, E.C., 2000. Nanoscale Elastic Properties of Dislocations and Disclinations. in: Nanostructured Films and Coatings, NATO Science Series Volume 78, Editors: G.-M. Chow, I.A. Ovidko, T. Tsakalakos, Kluwer, Dordrecht, pp 247–254.

Gutkin, M.Yu., Ovid'ko, I., 2004. Plastic Deformation in Nanocrystalline Materials. Springer, Berlin.

Gutkin, M.Yu., 2006. Elastic behaviour of defects in nanomaterials I.

Reviews on Advanced Materials Science **13**, 125–161.
Hassani, S., 2002. Mathematical Physics: A Modern Introduction to Its Foundations. Springer, Corr. 3rd printing.
Isaksson, P., Dumont, P.J.J., Rolland du Roscoat, S., 2012. Crack growth in planar elastic fiber materials. International Journal of Solids and Structures **49**, 1900–1907.
Isaksson, P., Hägglund, R., 2013. Crack-tip fields in gradient enhanced elasticity. Engineering Fracture Mechanics **97**, 186–192
Isaksson, P., Dumont, P.J.J., 2014. Approximation of mode I crack-tip displacement fields in gradient enhanced elasticity. Engineering Fracture Mechanics **117**, 1–11.
Jaunzemis, W., 1967. Continuum Mechanics. The Macmillan Company, New York.
Kanwal, R.P., 1983. Generalized Functions: Theory and Technique. Academic Press, New York.
Karlis, G.F., Tsinopoulos, S.V., Polyzos, D., Beskos, D.E., 2007. Boundary element analysis of mode I and mixed mode (I and II) crack problems of 2-D gradient elasticity. Computer Methods in Applied Mechanics and Engineering **196**, 5092–5103.
Konstantopoulos, I., Aifantis, E.C., 2013. Gradient elasticity applied to a crack. Journal of the Mechanical Behavior of Materials **22**, 193–201 (published December 06, 2013).
Koshlyakov, N.S., Smirnov, N.M., Gliner, E.B., 1964. Differential Equations of Mathematical Physics. North Holland, Amsterdam.
Lazar, M., 2002a. An elastoplastic theory of dislocations as a physical field theory with torsion. Journal of Physics A: Mathematical and General **35**, 1983–2004.
Lazar, M., 2002b. Screw dislocations in the field theory of elastoplasticity. Annalen der Physik (Leipzig) **11**, 635–649.
Lazar, M., 2003a. A nonsingular solution of the edge dislocation in the gauge theory of dislocations. Journal of Physics A: Mathematical and General **36**, 1415–1437.
Lazar, M., 2003b. Dislocations in the field theory of elastoplasticity. Computational Materials Science **28**, 419–428.
Lazar, M., Maugin, G.A., 2005. Nonsingular stress and strain fields of dislocations and disclinations in first strain gradient elasticity. International Journal of Engineering Science **43**, 1157–1184.
Lazar, M., Maugin, G.A., Aifantis, E.C., 2005. On dislocations in a special class of generalized elasticity. Physica Status Solidi (B) **242**, 2365–2390.
Lazar, M., Maugin, G.A., 2006. Dislocations in gradient elasticity revisited. Proceedings of the Royal Society A **462**, 3465–3480.
Lazar, M., Kirchner, H.O.K., 2007. The Eshelby stress tensor, angular momentum tensor and dilatation flux in gradient elasticity. International Journal of Solids and Structures **44**, 2477–2486.
Lazar, M., Anastassiadis, C., 2009. The gauge theory of dislocations: static solutions of screw and edge dislocations. Philosophical Magazine **89**, 199–231.
Lazar, M., Agiasofitou, E., 2011. Screw dislocation in nonlocal anisotropic elasticity. International Journal of Engineering Science **49**, 1404–1414.
Lazar, M., 2012. Non-singular dislocation loops in gradient elasticity. Physics Letters A **376**, 1757–1758.
Lazar, M., 2013. The fundamentals of non-singular dislocations in the theory of gradient elasticity: Dislocation loops and straight dislocations. International Journal of Solids and Structures **50**, 352–362.
Lazar, M., 2014. On gradient field theories: gradient magnetostatics and gradient elasticity. Philosophical Magazine **94**, 2840–2874.
Lazar, M., Polyzos, D., 2014. On non-singular crack fields in Helmholtz type enriched elasticity theories. International Journal of Solids and Structures, doi: 10.1016/j.ijsolstr.2014.01.002 (published January 15, 2014).
Ma, H.M., Gao, X.-L., 2011. Strain gradient solution for a finite-domain Eshelby-type plane strain inclusion problem and Eshelby's tensor for a cylindrical inclusion in a finite elastic matrix. International Journal of Solids and Structures **48**, 44–55.
Markolefas, S.I., Tsouvalas, D.A., Tsamasphyros, G.I., 2008. Mixed finite element formulation for the general anti-plane shear problem, including mode III crack computations, in the framework of dipolar linear gradient elasticity. Computational Mechanics **45**, 1–16.
Mindlin, R.D., 1964. Micro-structure in linear elasticity. Archive for Rational Mechanics and Analysis **16**, 51–78.
Paulino, G.H., Fannjing, A.C., Chan, Y.-S., 2003. Gradient elasticity for mode III fracture in functionally graded materials– part I: crack perpendicular to the material gradation. ASME Journal of Applied Mechanics **70**, 531–542.
Papanicolopulos, S.-A., Zervos, A., 2010. Numerical solution of crack problems in gradient elasticity. Engineering and Computational Mechanics **163**, 73–82.
Peerlings, R.H.J., Geers, M.G.D., Borst, R. de, Brekelmans, W.A.M., 2001. A critical comparison of nonlocal and gradient-enhanced softening continua. International Journal of Solids and Structures **38**, 7723–7746.
Po, G., Lazar, M., Seif, D., Ghoniem, N., 2014. Singularity-free dislocation dynamics with strain gradient elasticity. Journal of the Mechanics and Physics of Solids **68**, 161–178.
Podolsky, B., 1942. A generalized electrodynamics: part I – non-quantum. Physical Review **62**, 68–71.
Polyanin, A.D., 2001. Handbook of Linear Partial Differential Equations for Engineers and Scientists. Chapman and Hall/CRC, Boca Raton.
Polyzos, D., Tsepoura, K.G., Tsinopoulos, S.V., Beskos, D.E., 2003. A boundary element method for solving 2-D and 3-D static gradient elastic problems. Part I: Integral formulation. Computer Methods in Applied Mechanics and Engineering **192**, 2845–2873.
Roos, B.W., 1969. Analytic Functions and Distributions in Physics and Engineering. John Wiley & Sons Inc, New York.
Ru, C.Q., Aifantis, E.C., 1993. A simple approach to solve boundary-value problems in gradient elasticity. Acta Mechanica **101**, 59–68.
Shi, M.X., Huang, Y., Hwang, K.C., 2000. Fracture in the higher-order elastic continuum. Journal of the Mechanics and Physics of Solids **48**, 2513–2538.
Stamoulis, K., Giannakopoulos, A.E., 2008. Size effects on strength, toughness and fatigue crack growth of gradient elastic solids. International Journal of Solids and Structures **45**, 4921–4935.
Unger, D.J., Aifantis, E.C., 1995. The Asymptotic Solution of Gradient Elasticity of Mode III. International Journal of Fracture **71**, R27–R32.
Vardoulakis, I., Sulem, J., 1995. Bifurcation Analysis in Geomechanics. Blackie Academic and Professional (Chapman and Hall), London.
Vardoulakis, I., Exadaktylos, G., Aifantis, E., 1996. Gradient elasticity with surface energy: Mode-III crack problem. International Journal of Solids and Structures **33**, 4531–4559.
Vekua, I.N., 1967. New Methods for Solving Elliptic Equations. North-Holland, Amsterdam.
Vladimirov, V.S., 1971. Equations of Mathematical Physics. Marcel Dekker, Inc., New York.
Zauderer, E., 1983. Partial Differential Equations of Applied Mathematics. John Wiley & Sons Inc, New York.